Learnability of E–stable Equilibria [1]

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First Submitted 27 May 2010

This Version May 2010

Abstract: In this paper, we propose a refinement of E–stability conditions that select equilibria more robust to specification of the learning algorithm within the RLS/SG/GSG class. We show that the mean–dynamics speed of convergence under RLS learning is an important component of such a refinement: E–stable equilibria that are characterized by a faster speed of convergence under RLS learning are more likely to be learnable under SG or GSG algorithms. An example of monetary policy under commitment, with a determinate and E-stable REE suggests that such equilibria may fail to imply learnability when private agents update

[1] We thank Chrissy Giannitsarou and John Duffy for the encouragement to pursue this project. We are also indebted to seminar participants at the University of Edinburgh, Heriot Watt University, and the University of Wisconsin-Milwaukee. An earlier draft of the paper was presented at CEF 2006. We are grateful to the participants of the Learning Week at the Federal Reserve Bank of St. Louis, July 2007, for their contributions and comments on the paper, and especially George Evans. The second author has benefited from funding by the Royal Society of Edinburgh. All remaining errors are our own.

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their belief with an alternative learning algorithm and the RLS speed of convergence is slow.

Journal of Economic Literature Classification Numbers: C62, D83, E17.

Keywords: Adaptive learning, Expectational Stability, Stochastic Gradient, Speed of Convergence

1. Introduction

The concepts of adaptive learning and expectational stability (E–stability) in macroeconomics have received a well-deserved attention recently. The literature on adaptive learning assumes that agents act as econometricians who run recursive regressions using historical data to inform their decisions. Adaptive learning is just one description of real-world decision making processes, and not only of its possible asymptotic outcome which could be consistent with rational expectations. Evans and Honkapohja (2001) provide the methodology and derive the conditions under which recursive learning dynamics converges to rational expectations equilibria. If the economic agents use recursive least squares (RLS) learning to update their expectations of the future (or learn adaptively), then only E–stable REE can be the asymptotic outcomes of a real–time learning process. Equilibria, stable under a particular form of adaptive learning, are also called *learnable*.[4] Thus, E–stability is a necessary condition for RLS learnability. Bullard (2006) and Bullard and Mitra (2007) propose that E–stability of the resulting REE is a minimum property of any monetary policy rule, in effect requiring equilibria to be learnable under RLS.

Evans and Honkapohja (2001) analysis also points to the lack of general results on convergence for different learning algorithms. Barucci and Landi (1997) and Heinemann (2000) demonstrate that E–stability may not be a sufficient condition for learnability if adaptive algorithms other than RLS

[4]The possible convergence of learning processes and the E–stability criterion of REE dates back to DeCanio (1979) and Evans (1985). Marcet and Sargent (1989) first showed the conditions for convergence in a learning context using stochastic approximation techniques.

are being used. Barucci and Landi (1997) show that an alternative learning mechanism, namely, stochastic gradient (SG) converges to REE but under different conditions than RLS learning. Furthermore, Giannitsarou (2005) provides some examples of E–stable equilibria which are not learnable under SG learning. Evans et al. (2010) discuss additional conditions (related to the knowledge of certain characteristics of the REE) under which E–stable equilibria are learnable under SG and Generalized Stochastic Gradient (GSG) learning.

Tetlow and von zur Muehlen (2009) pose a problem of policy design in a world where agents might learn using a misspecified model of the economy. They state that an equilibrium which is learnable for a wide range of possible specifications, even at a potential cost of welfare losses, is a valuable property of a monetary policy rule. In this paper, we focus on the properties of E–stable equilibria that might facilitate such a design problem by making learnability under both RLS and GSG more likely. We propose a refinement of E–stability conditions that select equilibria more robust to specification of the learning algorithm within the RLS/SG/GSG class. We show that the mean–dynamics speed of convergence under the RLS learning is an important component of such a refinement, because E–stable equilibria that are characterized by a faster RLS convergence speed are more likely to remain learnable under SG or GSG algorithms as well. The mean-dynamics convergence speed can also have consequences for welfare, discussed in Ferrero (2007), and is related to the asymptotic behavior of the agents' beliefs as demonstrated by Marcet and Sargent (1995).

The E–stability criterion is a minimum requirement for the design of a meaningful monetary policy rule, see Bullard and Mitra (2007).[5] In this paper, we extend E–stability requirements by allowing for two additional criteria. First, we require that REE be learnable under a broad set of learning algorithms of the RLS/SG/GSG class. In some sense, this allows us to choose a subset of REE with properties such that they remain learnable even when agents' learning process is misspecified asymptotically relative to the RLS. Second, the speed of convergence under RLS should be fast not only to aid learnability, but also to ensure a fairly quick return of agents' beliefs towards the REE even after a small disturbance or deviation.[6] We show that the two additional criteria are related and can be met simultaneously.

2. E–Stability and Learnability Revisited

It is well established in Evans and Honkapohja (2001) and elsewhere that the convergence of the RLS algorithm is closely related to E–stability. The equilibrium is said to be E–stable if a stationary point $\overline{\Phi}$ of the following ordinary differential equation (ODE) is asymptotically stable:

$$\frac{d\Phi}{d\tau} = T(\Phi) - \Phi. \tag{1}$$

$\overline{\Phi}$ corresponds to the rational expectations equilibrium of a forward–looking model. T is the mapping from perceived law of motion (PLM) to actual

[5]In the literature, the concepts of E–stability and learnability are used interchangeably. Following Giannitsarou (2005) we consider learnability as a broader concept and distinguish between E–stability that is related to learnability under RLS and learnability under different learning algorithms.

[6]Ferrero (2007) discusses the welfare consequences of slow adjustment of inflationary expectations to their REE values.

law of motion (ALM), and Φ is a vector of the parameters of interest. The differential equation (1) describes the behavior of approximating, or "mean" dynamics in continuous "notional" time.[7] Its equilibrium point is asymptotically stable if the Jacobian of (1) evaluated at $\overline{\Phi}$,

$$J = DT(\Phi)\,|_{\Phi=\overline{\Phi}} - I,$$

has only eigenvalues with negative real parts.[8]

If, instead of using RLS, economic agents rely on SG learning, the convergence of the mean dynamics of the learning process is governed by the following ODE,

$$\frac{d\Phi}{d\tau} = M(\Phi) \cdot (T(\Phi) - \Phi)\,, \tag{2}$$

where $M(\Phi)$ is a symmetric and positive-definite matrix of second moments of the state variables used by agents in forming their forecasts.

The RE equilibrium $\overline{\Phi}$ is still a stationary point of (2). It is learnable if $\overline{\Phi}$ is the locally asymptotically stable equilibrium of the ODE (2), which obtains when all eigenvalues of $M(\overline{\Phi}) \cdot J$ have negative real parts. Barucci and Landi (1997) first provided a proof of this result. It is important to note that the conditions which establish the analogue of the E–stability condition in this case are different from those obtained under RLS.

[7] There are technical conditions other than the stability of the approximating mean dynamics which ought to be satisfied for convergence of the real–time dynamics under learning, see Evans and Honkapohja (2001) Chapter 6, Sections 6.2 & 6.3. We assume that these conditions are always satisfied, and claim that learnability is obtained when the equilibrium is stable under the approximating mean dynamics.

[8] The exceptional cases where we observe eigenvalues with zero real part do not typically arise in economic models.

If the agents update their beliefs with a Generalized SG learning (GSG) algorithm instead, learnability is related to the negative real parts of all eigenvalues of the matrix

$$\Gamma M(\overline{\Phi}) \cdot J, \tag{3}$$

where Γ is an arbitrary positive definite matrix, and therefore $\Gamma M(\overline{\Phi})$ is also arbitrary positive definite. This fact is well documented and illustrated in Evans et al. (2010).

The problem of a correspondence between E–stability and learnability under GSG learning is, therefore, equivalent to the following linear algebraic problem. Given a matrix J with all its eigenvalues to the left of the imaginary axis, could we guarantee that no eigenvalue of $\Gamma M(\overline{\Phi}) \cdot J$ becomes positive? This problem is well known and is referred to as H–stability, and discussed earlier in Arrow (1974), Johnson (1974b), Johnson (1974a), and Carlson (1968). A sufficient condition for H–stability that is easy to check exists: matrix J is H–stable if its symmetric part, $\left(\frac{J+J^T}{2}\right)$, is stable. Such a matrix is called negative quasi–definite. It is rather difficult to interpret this condition meaningfully from an economic point of view. Evans et al. (2010) provide an economic example and an extended discussion of GSG learning and H–stability.

While the convergence of the adaptive learning algorithms has been extensively studied, the transition, along the learning path, towards the equilibrium REE of interest is less well understood. Our starting point of reference is the results in Benveniste et al. (1990) and Marcet and Sargent (1995) who first identified the behavior of the speed of convergence (how fast or slow agents' beliefs approach a REE point) and analyzed the asymptotic properties of the fixed point under RLS learning. The behavior along the transition path and the importance of short-run deviations away from the REE was illustrated by Evans and Honkapohja (1993) and Marcet and Sargent (1995). Ferrero (2007) further argued that the speed of convergence can be considered as an important policy variable in the design of good monetary policy. An open question is how important is the RLS speed of convergence for learnability under alternative learning algorithms.

For the linearized E–stability ODE

$$\frac{d\Phi}{d\tau} = J \cdot \Phi, \tag{4}$$

where all eigenvalues of J are distinct and have only negative real parts, the solution would be given as a linear combination of terms of the form $C_i \cdot e^{\lambda_i \cdot t}$, where λ_is are the eigenvalues of J and C_i are arbitrary constants. In the long run, the solution is dominated by the term which corresponds to the smallest $\mid \lambda_i \mid$. This value is often called the *speed of convergence.* In the context of adaptive learning, the speed of convergence determines how fast the approximating mean dynamics described by the ODE in (1) approaches the REE asymptotically.[9]

The concepts of learnability under GSG and the speed of convergence under RLS appear to be distinct and far apart. However, it turns out that there is a close connection between the two. Consider, for example, the model in Sections 2 and 3 of Giannitsarou (2005). The reduced form of this

[9]Note that the relationship between the actual discrete–time and the approximating continuous "notional time" is not linear; this makes the rate at which the learning dynamics subside time-varying.

univariate model is given by

$$y_t = \lambda y_{t-1} + \alpha E_t^* y_{t+1} + \gamma w_t,$$
$$w_t = \rho w_{t-1} + u_t.$$

In this model, $|\rho| < 1$ and $u_t \sim N(0, \sigma_u^2)$. The equilibrium of the model in Giannitsarou (2005), with the same parameter values as in the paper: $\gamma = \sigma_u = 0.5$, $\rho = 0.9$, is E–stable, and therefore learnable under RLS. Both eigenvalues are real for all values of (α, λ) for which the solution $\overline{\Phi}_-$ is stationary and E–stable.

The E–stability ODE for this model is given by equation (1), where the mapping T is defined by equation (4) of Giannitsarou (2005), and the vector $\mathit{\Phi}$ is two–dimensional. Figures 1 and 2 summarize the negative quasi–definiteness for the corresponding Jacobian and the speed of convergence of the mean dynamics, respectively, as a function of the parameters α and λ. These figures clearly indicate that negative quasi–definiteness obtains in regions in the parameter space where the convergence speed is higher. Figure 2 is a contour plot of the speeds of convergence which increase towards the lower left corner of the graph, where the negative quasi–definiteness of the Jacobian is also observed.

In the region of the parameter space where the Jacobian is not negative quasi–definite, we might expect that a matrix Γ exists such that $\Gamma M(\overline{\mathit{\Phi}}) \cdot J$ is not stable. Therefore, the GSG learning algorithm that corresponds to this Γ does not result in an approximating dynamics converging to the REE. This conjecture stands correct: Giannitsarou (2005) shows that the equilibrium achieved under the SG learning algorithm (for which Γ equals the identity matrix) could not be learned for a small set of parameter values. Figure 2 illustrates that for these parameter values the equilibrium is not learnable under SG, and the speed of convergence is close to zero. On the other hand, for parameter values that correspond to a negative quasi–definite Jacobian, the speed of convergence is large and is never less than 0.35. Given that the negative quasi–definiteness is a sufficient condition for H–stability, we are guaranteed that GSG learning with any choice of Γ induces a learnable equilibrium. In particular, SG–learning will always converge for this set of parameter values.

[Figure 1 and Figure 2 about here]

This analysis indicates that there appears to exist a close relationship between learnability and the mean-dynamics speed of convergence under RLS learning. This paper studies the nature of this relationship and addresses the following questions:

1. Why is the lack of learnability for certain learning algorithms associated with a lower speed of convergence under RLS?
2. In contrast, why do conditions which guarantee fast convergence to REE, also seem to ensure learnability for any SG and GSG learning algorithm?
3. Are the answers to (1) and (2) general and applicable enough to a wider variety of self–referential models?

In what follows, we provide a two–dimensional geometric interpretation of the case when E–stability holds but learnability is not achieved (i.e., a matrix J is stable but not H–stable), and relate this finding to the speed of convergence of expectations to their REE values under RLS learning. In other words, when does E–stability fail to imply learnability for some GSG learning algorithm?

3. A Geometric Interpretation of Learnability

Provided J is a stable matrix , when is $\Omega \cdot J$ stable? We propose a simple two-dimensional geometric approach to answer this question. To preview our results: we study the eigenvalues of a matrix J, but not its components, and then relate those values to the speed of convergence of the mean dynamics under recursive least-squares learning. These findings have an intuitive and meaningful interpretation in a wide variety of adaptive learning models.

Now suppose that the 2×2 matrix J has only eigenvalues with negative real parts. This matrix is the Jacobian of the ODE in (1) for some adaptive learning model. J is asymptotically stable, therefore, the equilibrium associated with the model is E–stable and learnable under RLS.

The eigenvalue problem of J can be written as

$$J \cdot V = V \cdot \Lambda, \tag{5}$$

where V is the matrix with columns containing the eigenvectors of J, and Λ is diagonal with the corresponding eigenvalues λ_i on the main diagonal. When the eigenvectors are linearly independent, the matrix J can be diagonalized as $J = V \Lambda V^{-1}$. Learnability of the equilibrium under GSG adaptive learning is determined by the eigenvalues of $\Omega \cdot J$, where Ω is assumed to be symmetric and positive definite and thus can be written as $\Omega = PDP^T$.[10] The eigenvalue

[10]The eigenvalues of a symmetric matrix are orthogonal, and so $P^{-1} = P^T$.

problem for $\Omega \cdot J$ can therefore be written as:

$$PDP^T \cdot V\Lambda V^{-1} \cdot \widetilde{V} = \widetilde{V} \cdot \widetilde{\Lambda}, \tag{6}$$

where the columns of $\widetilde{V}$ are the eigenvectors of $\Omega \cdot J$ and $\widetilde{\Lambda}$ is a diagonal matrix with the eigenvalues of $\Omega \cdot J$ as the main entries.

Next pre–multiply (6) by P^{-1} and define $\overline{V} = P^{-1}\widetilde{V}$ to get

$$D \cdot P^T V\Lambda V^{-1} P \cdot \overline{V} = D\widetilde{J} \cdot \overline{V} = \overline{V} \cdot \widetilde{\Lambda}. \tag{7}$$

It is clear that the matrix $\widetilde{J} = P^T V\Lambda V^{-1} P$ has the same eigenvalues as J, i.e., the values on the main diagonal of Λ. Geometrically, if J represents a linear mapping in a two–dimensional space, then $\widetilde{J}$ represents the same mapping in new coordinates, given by the two orthogonal eigenvectors of Ω.

We work in the new coordinates and replace the problem of seeking conditions on the eigenvalues and eigenvectors of J such that $\Omega \cdot J$ has a positive eigenvalue (i.e., becomes unstable) with the equivalent problem concerning $\widetilde{J}$ and $D\widetilde{J}$. To fix notation, let us order δ_1 and δ_2, the eigenvalues of Ω, so that the following is always true, $\frac{\delta_2}{\delta_1} > 1$. The eigenvalues of J and $\widetilde{J}$ are $-\lambda_1$ and $-\lambda_2$, and ordered so that $\frac{|\lambda_2|}{|\lambda_1|} > 1$.[11] Denote the eigenvectors of $\widetilde{J}$ corresponding to $-\lambda_1$ and $-\lambda_2$ as $v_1 = (v_{11}, v_{21})^T$ and $v_2 = (v_{12}, v_{22})^T$. Define $\Upsilon = \frac{v_{22}}{v_{21}} \frac{v_{11}}{v_{12}}$.

Proposition 1: *Let $\lambda_{1,2}$ be real. The matrix $\Omega \cdot J$ has a positive eigenvalue and thus J is not H–stable iff the following conditions hold:*

(i) $0 < \Upsilon < 1$,

[11] Given the notation, the speed of convergence is equal to $|\lambda_1|$.

(ii) $\frac{\lambda_2}{\lambda_1} > \frac{1}{\Upsilon}$ *, and*

(iii)

$$\frac{\delta_2}{\delta_1} > \frac{\left(\frac{\lambda_2}{\lambda_1} - \Upsilon\right)}{\left(\frac{\lambda_2}{\lambda_1}\Upsilon - 1\right)} \tag{8}$$

Proof. See Appendix A. □

COROLLARY 1: *Let $\lambda_{1,2}$ be real. If either $\Upsilon < 0$ or $\Upsilon > \frac{\lambda_1}{\lambda_2}$, the matrix J is H–stable and the equilibrium is learnable for any GSG learning algorithm.*

Proof. See Appendix A. □

Proposition 1 constructs a counterexample of a matrix Ω such that $\Omega \cdot J$ is not stable. This means that agents who update their beliefs adaptively with the corresponding GSG algorithm cannot learn the REE, even though it is E–stable. A necessary condition for E–stable REE not to be learnable under some GSG learning algorithm is a positive Υ less than one. Geometrically, $0 < \Upsilon < 1$ implies that, after rotation into the system of coordinates defined by the eigenvectors of Ω, the two eigenvectors of J are in the same quadrant. When the eigenvectors are close to being collinear, finding Ω such that it satisfies this geometric condition is more likely.[12]

It might be impossible to satisfy the assumptions of Proposition 1 if the eigenvalues are close to being orthogonal, as in this case Υ is either too small (less than $\frac{\lambda_1}{\lambda_2}$) or too large (above 1) when it is positive. If they are exactly orthogonal the positive $\Upsilon = 0$ or ∞. The necessary (and sufficient)

[12]Note that the angle between the eigenvectors of J is preserved under the rotation into the orthogonal coordinate system determined by the eigenvectors of Ω. Therefore, we use collinearity of the eigenvectors of J and $\widetilde{J}$ interchangeably.

condition described in (8) shows that, for a given Υ, instability of $\Omega \cdot J$ is more likely when either $\frac{\lambda_2}{\lambda_1}$ is large or $\frac{\delta_2}{\delta_1}$ is large or both. The latter occurs when the eigenvalues of Ω are highly unbalanced. If the agents update their beliefs using SG learning, for example, Ω is the second-moments matrix of regressors with highly unequal diagonal terms.

The ratio $\frac{\lambda_2}{\lambda_1}$ will be large if $|\lambda_1|$ is very small, or in other words, the speed of convergence under RLS is small. Increasing $|\lambda_1|$ will facilitate learnability as the condition in Proposition 1 becomes more difficult to fulfill. In particular, it will require much more unbalanced matrix Ω. The economic agents, who learn adaptively, are less likely to be using SG algorithm when the variances of the regressors are extremely unequal. Therefore, the speed–of–convergence criterion directly ensures that the set of equilibria learnable by agents using algorithms within the RLS/SG/GSG class is sufficiently large.

Figure 3 plots the necessary and sufficient condition (8) for three values of Υ: 0.3, 0.1 and 0.05 (eigenvectors of $\widetilde{J}$ close to being orthogonal), respectively. The condition is satisfied in the area of the figure located above and to the right of the corresponding line. If v_1 and v_2 are almost collinear, instability (and therefore, the lack of learnability) could be achieved for relatively mild ratios of the eigenvalues $\frac{\lambda_2}{\lambda_1}$ and $\frac{\delta_2}{\delta_1}$.

[Figure 3 about here]

Turning to the results of Giannitsarou (2005), we show that points in the parameter space for which the equilibrium is not learnable under SG satisfy the conditions of Proposition 1, as the corresponding values of $\frac{\lambda_2}{\lambda_1}$ and $\frac{\delta_2}{\delta_1}$ are extreme, and lie in the vicinity of 112 and $5.5 \cdot 10^5$, respectively. Such a high degree of imbalance in the matrix Ω is explained by observing that Υ is very close to zero (the eigenvectors of J are almost orthogonal) throughout the whole parameter space. For these parameters the speed of convergence is slow (in the order of magnitude of 10^{-3}), and hence the ratio of the eigenvalues of J is very large.

For completeness, we next turn to the case when the eigenvalues of the stability matrix are complex. To fix notation, assume that J has two complex eigenvalues: $\nu \pm i\mu$, $\nu < 0$, and two complex eigenvectors $w_1 \pm iw_2$, where w_1 equals $(w_{11}, w_{21})'$ and w_2 is $(w_{12}, w_{22})'$. Define $\widetilde{W} = w_{11}w_{12} + w_{21}w_{22}$ and $|W| = w_{11}w_{22} - w_{12}w_{21}$. The following Proposition provides the necessary and sufficient condition for the instability of the matrix $\Omega \cdot J$:

Proposition 2: *Let $\lambda_{1,2}$ be complex. The matrix $\Omega \cdot J$ has a positive eigenvalue and thus J is not H–stable iff the following condition holds:*

$$\frac{\mu}{|\nu|}\frac{\widetilde{W}}{|W|}\frac{\left(\frac{\delta_2}{\delta_1}-1\right)}{\left(\frac{\delta_2}{\delta_1}+1\right)} > 1. \tag{9}$$

Proof. See Appendix B. □

Similarly, Proposition 2 demonstrates that a smaller ratio of the eigenvalues of Ω is conducive to the stability of $\Omega \cdot J$. Hence, the corresponding GSG learning algorithm generates a convergent dynamics. A higher speed of convergence, larger $|\nu|$, makes the necessary and sufficient condition described in (9) harder to fulfill, and thus increases the set of parameters for which equilibria are learnable. Orthogonality of w_1 and w_2 means that the condition in Proposition 2 cannot be satisfied. Notice a similarity between the results for real and complex eigenvalues. In both cases, orthogonality of eigenvectors (real case) or their real and imaginary components (complex case) ensures that learning instability is impossible.

Propositions 1 and 2 indicate that the second criterion we impose on all desirable REE — the high speed of convergence under RLS learning — is in accordance with the first criterion, namely the REE are learnable under a range of learning algorithms within the RLS/SG/GSG class. To illustrate further the alignment of these two criteria and the way in which they modify selection of monetary policy rules, we study a standard model of monetary policy under commitment with learning.

4. Monetary Policy under Commitment

4.1. The model environment

Following Evans and Honkapohja (2006), we start with a standard two-equation New Keynesian (NK) model:

$$x_t = -\varphi\left(i_t - \widehat{\pi}_{t+1}\right) + \widehat{x}_{t+1}, \tag{10a}$$

$$\pi_t = \lambda x_t + \beta \widehat{\pi}_{t+1} + u_t. \tag{10b}$$

Here x_t and π_t express the output gap and inflation in period t, and all variables with hats denote private sector expectations. i_t is the nominal interest rate, in deviation from its long run steady-state. The parameters φ and λ are positive and have the standard explanation and the discount factor is $0 < \beta < 1$. Our main interest is in the learning behavior of private sector agents, and we maintain the assumption that expectations may not be rational. We also assume the presence of only one shock to illustrate the results in this paper, and disregard the influence of the demand shock in the IS equation (10). The cost-push shock in (10b) is given by $u_t = \rho u_{t-1} + \epsilon_t$ where $\epsilon_t \sim iid(0, \sigma_\epsilon^2)$ is independent white noise. In addition, $|\rho| < 1$.

We consider the expectations–based interest rate policy rule under commitment, using timeless perspective solution:

$$i_t = \phi_L x_{t-1} + \phi_\pi \widehat{\pi}_{t+1} + \phi_x \widehat{x}_{t+1} + \phi_u u_t. \tag{11}$$

The optimal values of the policy rule parameters, based on a standard loss function, are given in Evans and Honkapohja (2006), Eq. (15) (notice that the coefficient ϕ_g is assumed to be zero in our specification). Here we do not restrict our attention to optimal monetary policy. We fix the values of the policy parameters, ϕ_L and ϕ_u, at their optimal level, and treat the other two policy parameters as choice variables of the policy response of the monetary authority.

Under the assumed policy rule the model can be written as:

$$\begin{aligned} y_t = \begin{bmatrix} x_t \\ \pi_t \end{bmatrix} &= \begin{bmatrix} 1 - \varphi\phi_x & \varphi(1-\phi_\pi) \\ \lambda(1-\varphi\phi_x) & \lambda\varphi(1-\phi_\pi) + \beta \end{bmatrix} \begin{bmatrix} \widehat{x}_{t+1} \\ \widehat{\pi}_{t+1} \end{bmatrix} \\ &+ \begin{bmatrix} -\varphi\phi_L & 0 \\ -\varphi\lambda\phi_L & 0 \end{bmatrix} \begin{bmatrix} x_{t-1} \\ \pi_{t-1} \end{bmatrix} + \begin{bmatrix} -\varphi\phi_u \\ 1 - \lambda\varphi\phi_u \end{bmatrix} u_t \qquad (12) \\ y_t &= A\widehat{y}_{t+1} + CJ^T y_{t-1} + Bu_t \end{aligned}$$

where $J = (1, 0)^T$.

The MSV solution of this system can be expressed in the following way, with c and b both being vectors such as $c = (c^x, c^\pi)^T$, $b = (b^x, b^\pi)^T$:

$$y_t = cJ^T y_{t-1} + bu_t. \tag{13}$$

Using the method of undetermined coefficients we find the REE solution, where c^x solves the cubic equation:

$$c^x = -\varphi\phi_L + (1-\varphi\phi_x)(c^x)^2 + \frac{\lambda\varphi(1-\phi_\pi)(c^x)^2}{1-\beta c^x}.$$

The rest of the solution is provided in:

$$c^\pi = \frac{\lambda c^x}{1-\beta c^x}$$

and

$$b = \left(I - A\left(cJ^T + \rho I\right)\right)^{-1} B.$$

Next we turn to the conditions under which REE are learnable. We check for determinacy of the RE solution using the conditions derived in Evans and Honkapohja (2006), page 35, which are not reproduced here.

4.2. Determinacy and E–stability: the minimum requirements for desirable RE equilibria

Discussing E–stability, we follow the rest of the literature in assuming that the MSV solution obtained above is the PLM used by the private sector agents in the model. Let us re-write the model as:

$$\begin{aligned} y_t &= A\widehat{y}_{t+1} + CJ^T y_{t-1} + Bu_t, \\ u_t &= \rho u_{t-1} + \epsilon_t. \end{aligned}$$

Calculate $\widehat{y_{t+1}}$, the non-rational expectations of the model (13), as:

$$\widehat{y}_{t+1} = E_t^*[cJ^T y_t + bu_{t+1}] \;\; = \left[A\left(cJ^T\right)\left(cJ^T\right) + CJ^T\right] y_{t-1} + \left[A\left(cJ^T + \rho I\right) b + B\right] u_t.$$

Therefore, the T–map for the problem becomes:

$$T(b,c) = \left(A\left(cJ^T + \rho I\right)b + B, \left(c^T J\right)Ac + C\right).$$

This allows us to compute the (4×4) Jacobian matrix:

$$J = \begin{bmatrix} A\left(\bar{c}J^T + \rho I\right) - I_2 & \left(J^T \bar{b}\right)A \\ 0 & A\left(\bar{c}J^T + J^T\bar{c} \cdot I\right) - I_2 \end{bmatrix}.$$

4.3. SG–stability

The matrix, M_z, of the second moments of the state variables in the model used by the agents to forecast the inflation and the output gap is obtained from a two–variable VAR written as:

$$\begin{bmatrix} x_t \\ u_{t+1} \end{bmatrix} = \begin{bmatrix} c^x & b^x \\ 0 & \rho \end{bmatrix} \begin{bmatrix} x_{t-1} \\ u_t \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \epsilon_{t+1}.$$

Then, to study the SG–stability of the model, we examine the eigenvalues of $\Omega \cdot J$ which now is:

$$\Omega \cdot J = \left(M_z \otimes I_2\right) J.$$

We also study the GSG–stability assuming that the agents are updating their beliefs about the parameters in the model by making small errors around the outcomes under RLS learning. They adopt Γ in (3) to be the inverse of a second moments matrix $\widetilde{M_z}$, which is compatible with the same model evaluated at parameter values obtained within the neighborhood of the calibrated parameters used in the simulation analysis.

We assume alternative learning algorithms to model the agents' uncertainty about the second moments of the state variables which they need to know in order to be able to run regressions which make GSG learning asymptotically equivalent to RLS learning. Learnability of the equilibria under this assumption is achieved when all eigenvalues of the matrix $((\widetilde{M}_z^{-1} \cdot M_z) \otimes I_2) \cdot J$ have negative real parts.

Propositions 1 and 2 cannot be stated in the four dimensional case we have specified in the model, and therefore we analyze the stability of the system by simulations. However, we expect that the main findings presented in the Propositions remain valid in the higher dimensions as well, namely that the lower speed of convergence of the mean dynamics under RLS would be associated with higher incidence of instability of GSG learning.

5. Learning Instability and Equilibria: Discussion

To analyze further the link between learnability under GSG and the speed of convergence under RLS, we resort to simulations of the simple NK model under commitment for different values of the expectations-based policy rule parameters. To be more precise, we keep the parameters ϕ_L and ϕ_u at their optimal values derived in Evans and Honkapohja (2006), but vary ϕ_π and ϕ_x in a sufficiently broad range. The theoretical results on expectation-based policy rules under commitment, namely determinacy and E–stability of the REE, for any parameter values, were derived only for optimal policy by Evans and Honkapohja (2006). Therefore, we proceed to check every point for determinacy and E–stability (i.e., checking the eigenvalues of J for a negative real part). In addition, we calculate the speed of convergence of the mean dynamics under RLS learning, as described in Section 2, and check for convergence of the SG learning, by evaluating the eigenvalues of $(M_z \otimes I_2) \cdot J$.

We calibrate our model using the following parameter values. They are the same as in Richard Clarida and Gertler (1999) calibration: $\beta = 0.99$, $\varphi = 4$ and $\lambda = 0.075$. We assume different values for the persistence of cost–push shock, $\rho = 0.90$ (commonly used in the literature) and $\rho = 0.60$.

The results of our simulations with $\rho = 0.90$ are presented in Figure 4. It illustrates, for every pair of the policy parameters (ϕ_π, ϕ_x), whether the resulting REE is determinate, E–stable, and SG–stable.[13] We plot only the points that are E–stable. The black area represents all indeterminate equilibria. We see that a standard Taylor principle applies (see Llosa and Tuesta (2009) for the theoretical derivations). The points satisfying Taylor principle are further split into SG–stable (white area) and SG–unstable (grey area).[14] SG–instability is concentrated in areas where ϕ_π is relatively low; as evident, more active monetary policy is associated with SG–stability.

How does this result fare against our Propositions 1 and 2, which associate the robustness of learnability under alternative algorithms with the higher convergence speed under RLS? They are vindicated fully, as Figure 5 confirms. This association is shown by plotting contour levels of convergence speeds for the same values of (ϕ_π, ϕ_x). All SG–unstable points have a low convergence speed. Both the convergence speed and the robustness of learnability increase as we move towards more active monetary policy under commitment.

[13]We also check the sufficient condition for H–stability (symmetric part of J stable) but in the range of our calibration and policy parameters no points were found to satisfy the condition. This indicates how restrictive the negative quasi-definiteness of a matrix proves to be.

[14]We do not track SG–stability for indeterminate REEs.

To compare our results with those of Evans and Honkapohja (2006), we plot a black asterisk at the point corresponding to the optimal monetary policy for our calibrated values.[15] As expected, this policy delivers determinate and E–stable REE; however, notice that this policy is very close to both SG–stability and E–stability boundaries. This proximity raises an issue of robustness of the optimal monetary policy if the agents are making small mistakes in their learning process.

We perform the following experiment to study the robustness of the optimal or near–optimal monetary policy. We assume that the agents update their beliefs by running not RLS but use a GSG learning algorithm instead. If the agents knew exactly the second moments matrix M_z associated with the parameter values (including the optimal monetary policy parameters) of the model, they would run a GSG that used M_z^{-1} as a weighting matrix. This GSG algorithm would be asymptotically equivalent to RLS, delivering determinate and E–stable REE.

In the experiment, we also assume that the agents face uncertainty regarding the second moments matrix. Given that the agents are *learning* second–order moments, such as correlations between future inflation and past output gap and the cost–push shock (and, therefore, they do not know them, at least away from the REE achieved asymptotically in infinite time), it seems natural to assume that their knowledge of other second–order moments is limited as well. We thus assume that the agents take the deep parameters of the model to be somewhere in the neighborhood of the "true" parameter vector

[15]To derive the optimal policy values used in our simulations, we assume a relative weight of 0.02 on the output gap.

θ that we use in simulations. The agents would believe in $\widetilde{\theta}$ and use it to compute the second moments matrix $\widetilde{M}_z(\widetilde{\theta})$.[16] Then the agents would tend to use the matrix $\widetilde{M}_z^{-1}$ as a weighting matrix in their updating of beliefs. Hence, $\widetilde{M}_z^{-1} = \Gamma$ in equation (3). The resulting condition for the convergence of this real–time learning process is given by all eigenvalues of the matrix $((\widetilde{M}_z^{-1} \cdot M_z) \otimes I_2) \cdot J$ being negative.

We draw realizations of agents' beliefs about the parameters, $\widetilde{\theta}$, from a distribution that is centered at true parameters θ. The range of the distribution is comparable to the prior distributions usually found in the literature on estimated DSGE models, see, for example, Milani (2007). We nest the true RLS learning in this procedure because one could argue that SG–learning is too different from the RLS (for example, it is not scale invariant, see Evans et al. (2010)) to be a realistic description of any actual updating process. Then, we check whether this GSG algorithm is learnable or not. By repeating this procedure one thousand times, we obtain an estimate of probability of obtaining GSG instability for a given parameter pair, (ϕ_π, ϕ_x). [17,18]

The outcome of this simulation exercise, once again, confirms the results

[16]We do not believe that adaptively learning agents "knowing" exactly a wrong second-moments matrix is an assumption that is any more realistic than assuming that they are endowed with perfect knowledge of M_z. By using this procedure, we intend to generate "perturbed" second-moments matrices that have correlation structure that is similar to that of the true one.

[17]We assume that the agents keep parameters (ϕ_π, ϕ_x) the same but re–calculate (ϕ_L, ϕ_u).

[18]The exact values of probability of obtaining a GSG algorithm that delivers a learnable equilibrium depends on the assumed distribution of $\widetilde{\theta}$. We are only interested in the direction in the parameter space in which this probability increases or decreases.

described in Propositions 1 and 2. Figure 6 shows that, for the SG–unstable points with a low RLS mean-dynamics convergence speed, we generally observe high incidence of learning instability of GSG processes with imperfect knowledge of second moments matrix. For the lowest speeds still consistent with determinate and E–stable REE, we observe up to 60% probability of GSG instability. The optimal monetary policy (black asterisk) is associated with about 20% probability of becoming unlearnable when the agents use an algorithm other than RLS.

The probability of observing a divergent GSG algorithm measures only how likely it is to find parameter draws such that the agents' misperceptions become strong enough to lead to expectational instability. How far these mistaken perceptions should be from the "truth" in order to generate a divergent algorithm? To answer this question, we take the matrix $\widetilde{M}_z^{-1} M_z$ and evaluate its largest eigenvalue.[19] We take this value as a measure of the mismatch between the "true" second moments matrix M_z and the agents' erroneous beliefs $\widetilde{M}_z$. We further consider the minimum of this measure over those among the 1000 realizations that lead to divergent GSG algorithms, and plot their contour levels in Figure 7. The lower intensity of grey depicted to the right area of the figure are associated with the higher coutour levels. For example, the white area represents the strongest misperceptions about the true second-moments matrix. The darkest area corresponds to the least mistmatch of perceptions that allow divergent GSG algorithm.[20]

[19]When the agents use RLS learning, this eigenvalue is equal to one.

[20]This exercise is in the spirit of Tetlow and von zur Muehlen (2009) who model agents running RLS but making errors in the ALM. They study the minimum pertubation such that the resulting algorithm is divergent.

The results of this exercise point in the same direction as Figures 4 through 6. The more active the monetary policy is, the harder it is to generate a divergent learning algorithm, because the necessary mismatch of beliefs is stronger (the lighter areas in Figure 7). For the points in the (ϕ_π, ϕ_x) space which correspond to almost zero probability of observing GSG–unstable algorithm, the mismatch measure equals 10 or higher, as the few unstable ones require a very large mismatch of beliefs.[21]

The simulation results depicted in Figures 4 through 7 are consistent with the conditions established in Propositions 1 and 2. The faster the convergence of RLS adaptive learning mean dynamics is, the harder it is to generate a divergent GSG-type algorithm. Fast convergence speeds correspond to stability under SG-learning, lower probability of finding a second moments matrix, $\widetilde{M}_z$ and bigger misperceptions that is required to generate such a divergent GSG algorithm. Thus, the two 'refinements' to the concept of E-stability that we propose, the faster speed of RLS mean dynamics convergence and the greater robustness within a class of RLS/SG/GSG learning algorithms, do not need to generate trade-offs and might be both satisfied simultaneously.

We summarize the results for $\rho = 0.9$ in Table 1. We also perform some sensitivity analysis of our simulation results. Table 2 presents the results for the same set of parameter values as previously, but $\rho = 0.6$. In the presence of a highly persistent cost-push shock, a temporary increase in inflation

[21]In the simple case of a constant and iid shock, the value of 10 corresponds to the agents perceiving the shock as being 10 times more volatile than in reality, which is indeed a severe misperception.

might result in increased inflation expectations which would remain elevated for a prolonged period of time, induce actual inflation persistence as well. In other words, the convergence speed under adaptive learning decreases as the persistence of the shocks goes up, and thus any initial deviation in either actual inflation or inflation expectations takes longer to die out. When the shock persistence is *lower* than in the baseline model calibration, in accordance with Propositions 1 and 2 and the results of this section, we expect higher RLS adaptive learning mean-dynamics convergence speeds to lead to larger area of SG–stability, and lower probability of finding a GSG-unstable algorithms.

In Tables 1 and 2 the convergence speed is a decreasing function of the cost-push shock persistence.[22] The area of SG-instability disappears completely for values of persistence less than 0.6 (Table 2, column 3). For $\rho = 0.6$, the probability of GSG-stability is everywhere above 0.99, and it becomes essentially unity for the monetary policy with $\phi > 1.2$, which is less active than the optimal under the RE. (In the baseline calibration with persistence of 0.9, only a Taylor rule with ϕ_π as high as 3.5 guarantees GSG-stability).[23]

6. Conclusion

While under recursive least-squares learning the dynamics of linear and some nonlinear models converge to E–stable rational expectations equilibria, see Evans and Honkapohja (2001), some recent examples indicate that

[22]Compare the entries for the same ϕ_π and ϕ_x in Tables 1 and 2.

[23]The figures which illustrate the sensitivity to the shock persistence are not reported here and are available upon request from the authors.

E–stability is not a sufficient condition for learnability. In this paper, we establish that there is a close relationship between the learnability of E–stable equilibria and the speed of convergence of the RLS learning algorithm. In the 2x2 case, we provide conditions which ensure that fast mean-dynamics speed of convergence implies learnability under a broad set of learning algorithms of the RLS/SG/GSG class. In some sense, this is a refinement of the set of E–stable REE with properties such that learnability is achieved even when agents' learning is misspecified asymptotically relative to the RLS.

In addition, we quantify the significance of the RLS speed of convergence for learnability under alternative learning algorithms. **?** show that optimal monetary policy under commitment leads to expectational stability in private agents' learning. We provide evidence that such an E–stable RE equilibrium might fail to establish its generalized SG learnability when agents have misperceptions. For the lowest speeds of convergence consistent with determinate and E–stable REE, we observe up to 60% probability of GSG instability. The optimal monetary policy under commitment is also associated with approximately 20% probability of being subject to expectational instability when the agents use an algorithm other than RLS.

APPENDIX

A. Learning Instability: The Real Eigenvalues Case

First we examine the real eigenvalues case. We investigate the conditions under which $D\widetilde{J}$ has a positive eigenvalue, and is therefore unstable. Since $\left|D\widetilde{J}\right| = |D| \cdot \left|\widetilde{J}\right| > 0$, instability can only appear if the trace of $D\widetilde{J}$, denoted by $Tr(D\widetilde{J})$, is strictly positive. Write the matrix of the eigenvectors of $\widetilde{J}$ and its inverse[24] as

$$V = \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix}, \; V^{-1} = \frac{1}{|V|} \begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix},$$

then the matrix $\widetilde{J}$ can be written as:

$$\widetilde{J} = V \Lambda V^{-1} = \frac{1}{|V|} \begin{bmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{bmatrix} \cdot \begin{bmatrix} -\lambda_1 & 0 \\ 0 & -\lambda_2 \end{bmatrix} \cdot \begin{bmatrix} v_{22} & -v_{12} \\ -v_{21} & v_{11} \end{bmatrix}.$$

We can establish that the diagonal elements of $\widetilde{J}$ are given by:

$$\widetilde{J}_{11} = \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|V|}$$

and

$$\widetilde{J}_{22} = \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|V|}$$

.

The trace of the $D\widetilde{J}$ is thus equal to:

$$Tr(D\widetilde{J}) = \delta_1 \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|V|} + \delta_2 \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|V|} > 0. \qquad \text{(A.1)}$$

[24] The case of a non–invertible V is not generic, and we do not consider it here.

The condition (A.1) is equivalent to:

$$\delta_1 \frac{\lambda_2 v_{12} v_{21} - \lambda_1 v_{11} v_{22}}{|V|} + \delta_2 \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|V|} > 0$$
$$\frac{\delta_2}{\delta_1} \frac{\lambda_1 v_{12} v_{21} - \lambda_2 v_{11} v_{22}}{|V|} - \frac{\lambda_1 v_{11} v_{22} - \lambda_2 v_{12} v_{21}}{|V|} > 0$$
$$\frac{\delta_2}{\delta_1} \frac{v_{12} v_{21} - \frac{\lambda_2}{\lambda_1} v_{11} v_{22}}{|V|} - \frac{v_{11} v_{22} - \frac{\lambda_2}{\lambda_1} v_{12} v_{21}}{|V|} > 0.$$

Now select the direction of the eigenvectors so that $v_{12} v_{21} > 0$, denote $\Upsilon = \frac{v_{22}}{v_{21}} \frac{v_{11}}{v_{12}} = \frac{v_{22}}{v_{12}} / \frac{v_{21}}{v_{11}}$, and observe that $|V|$ equals $v_{21} v_{12} (\Upsilon - 1)$. The trace condition (A.1) becomes

$$\frac{\frac{\delta_2}{\delta_1} v_{12} v_{21}}{v_{21} v_{12} (\Upsilon - 1)} \left(1 - \frac{\lambda_2}{\lambda_1} \Upsilon\right) - \frac{v_{12} v_{21}}{v_{21} v_{12} (\Upsilon - 1)} \left(\Upsilon - \frac{\lambda_2}{\lambda_1}\right) > 0$$

or

$$\frac{1}{\Upsilon - 1} \left[\frac{\delta_2}{\delta_1} \left(1 - \frac{\lambda_2}{\lambda_1} \Upsilon\right) - \left(\Upsilon - \frac{\lambda_2}{\lambda_1}\right)\right] > 0. \qquad \text{(A.2)}$$

When $\Upsilon < 0$, this expression is clearly negative. Thus, learning instability requires that both eigenvectors of J after rotation into the coordinates defined by the eigenvectors of Ω are located in the same quadrant of the plane. This condition is impossible to meet if the two eigenvectors are orthogonal.

When $\Upsilon > 1$, the term in the square brackets in (A.2) is negative: it is a decreasing function of $\frac{\lambda_2}{\lambda_1}$, $\frac{\delta_2}{\delta_1}$, and Υ, reaching its maximum of 0 for $\frac{\lambda_2}{\lambda_1} = \frac{\delta_2}{\delta_1} = \Upsilon = 1$. Therefore, the whole expression (A.2) is negative for $\Upsilon > 1$.

When $0 < \Upsilon < 1$, learning instability requires

$$\frac{\delta_2}{\delta_1} \left(1 - \frac{\lambda_2}{\lambda_1} \Upsilon\right) < \Upsilon - \frac{\lambda_2}{\lambda_1} < 0.$$

This is possible only if $1-\frac{\lambda_2}{\lambda_1}\Upsilon < 0$ or $\frac{\lambda_2}{\lambda_1} > \frac{1}{\Upsilon} > 1$, in which case the condition above can be rewritten as

$$\frac{\delta_2}{\delta_1} > \frac{\Upsilon - \frac{\lambda_2}{\lambda_1}}{1 - \frac{\lambda_2}{\lambda_1}\Upsilon} = \frac{\frac{\lambda_2}{\lambda_1} - \Upsilon}{\frac{\lambda_2}{\lambda_1}\Upsilon - 1}.$$

B. Learning Instability: The Complex Eigenvalues Case

In this case, the eigenvalues of $\widetilde{J}$ are given by $\nu \pm i\mu$, $\nu < 0$, and the corresponding eigenvectors are $w_1 \pm iw_2$. Following the same steps as in the real roots case, write

$$\begin{aligned}
W &= \begin{bmatrix} w_{11} + iw_{12} & w_{11} - iw_{12} \\ w_{21} + iw_{22} & w_{21} - iw_{22} \end{bmatrix} \\
W^{-1} &= \frac{1}{2\,|W|} \begin{bmatrix} w_{22} + iw_{21} & -\left(w_{12} + iw_{11}\right) \\ w_{22} - iw_{21} & -\left(w_{12} - iw_{11}\right) \end{bmatrix} \\
|W| &= w_{11}w_{22} - w_{12}w_{21},
\end{aligned}$$

$$\begin{aligned}
W\Lambda W^{-1} = \frac{1}{2\,|W|} &\begin{bmatrix} w_{11} + iw_{12} & w_{11} - iw_{12} \\ w_{21} + iw_{22} & w_{21} - iw_{22} \end{bmatrix} \cdot \begin{bmatrix} \nu + i\mu & 0 \\ 0 & \nu - i\mu \end{bmatrix} \cdot \\
&\begin{bmatrix} w_{22} + iw_{21} & -\left(w_{12} + iw_{11}\right) \\ w_{22} - iw_{21} & -\left(w_{12} - iw_{11}\right) \end{bmatrix} \\
= \frac{1}{2\,|W|} &\begin{bmatrix} \left(\nu + i\mu\right)\left(w_{11} + iw_{12}\right) & \left(\overline{\nu + i\mu}\right)\left(\overline{w_{11} + iw_{12}}\right) \\ \left(\nu + i\mu\right)\left(w_{21} + iw_{22}\right) & \left(\overline{\nu + i\mu}\right)\left(\overline{w_{21} + iw_{22}}\right) \end{bmatrix} \cdot \\
&\begin{bmatrix} w_{22} + iw_{21} & -\left(w_{12} + iw_{11}\right) \\ \overline{w_{22} + iw_{21}} & -\left(\overline{w_{12} + iw_{11}}\right) \end{bmatrix}.
\end{aligned}$$

The overline in the expressions denotes a complex conjugate. Finally, the two diagonal elements of $\widetilde{J}$ can be written as:

$$\widetilde{J}_{11} = \frac{\mathrm{Re}\left[\left(\nu + i\mu\right)\left(w_{11} + iw_{21}\right)\left(w_{22} + iw_{21}\right)\right]}{|W|},$$
$$\widetilde{J}_{22} = -\frac{\mathrm{Re}\left[\left(\nu + i\mu\right)\left(w_{12} + iw_{22}\right)\left(w_{21} + iw_{11}\right)\right]}{|W|},$$

which reduces to

$$\widetilde{J}_{11} = \nu - \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|},$$
$$\widetilde{J}_{22} = \nu + \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}.$$

The trace of $D\widetilde{J}$ then is given by

$$Tr(D\widetilde{J}) = \nu\left(\delta_2 + \delta_1\right) + \mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}\left(\delta_2 - \delta_1\right) > 0 \qquad \text{(B.3)}$$

and should be positive for the instability to occur.

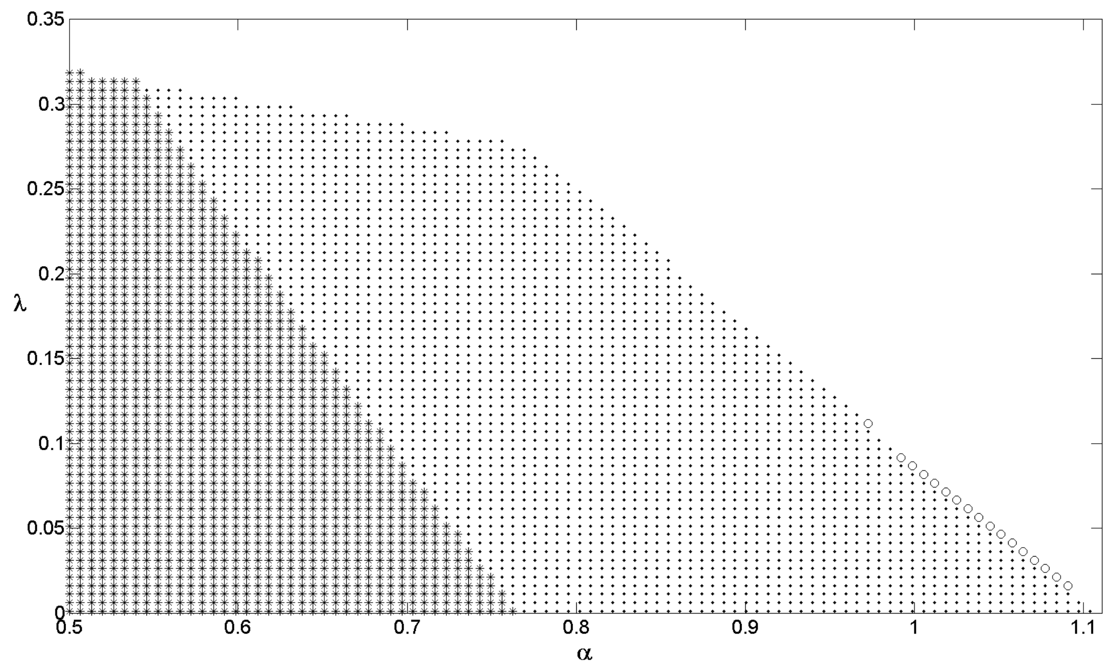
Let $\widetilde{W} = w_{11}w_{12} + w_{21}w_{22}$ and recall that ν is negative. Then (B.3) is equivalent to

$$\mu\frac{w_{11}w_{12} + w_{21}w_{22}}{|W|}\left(\delta_2 - \delta_1\right) > -\nu\left(\delta_2 + \delta_1\right),$$
$$\frac{\mu}{|\nu|}\frac{\widetilde{W}}{|W|}\frac{\frac{\delta_2}{\delta_1} - 1}{\frac{\delta_2}{\delta_1} + 1} > 1.$$

This expression allows us to evaluate and relate the speed of convergence, the real part of the eigenvalues in this case, and the conditions for learning instability. It is easy to show that if $w_1 \perp w_2$, then $\frac{\widetilde{W}}{|W|} = 0$ making (9) impossible to satisfy.

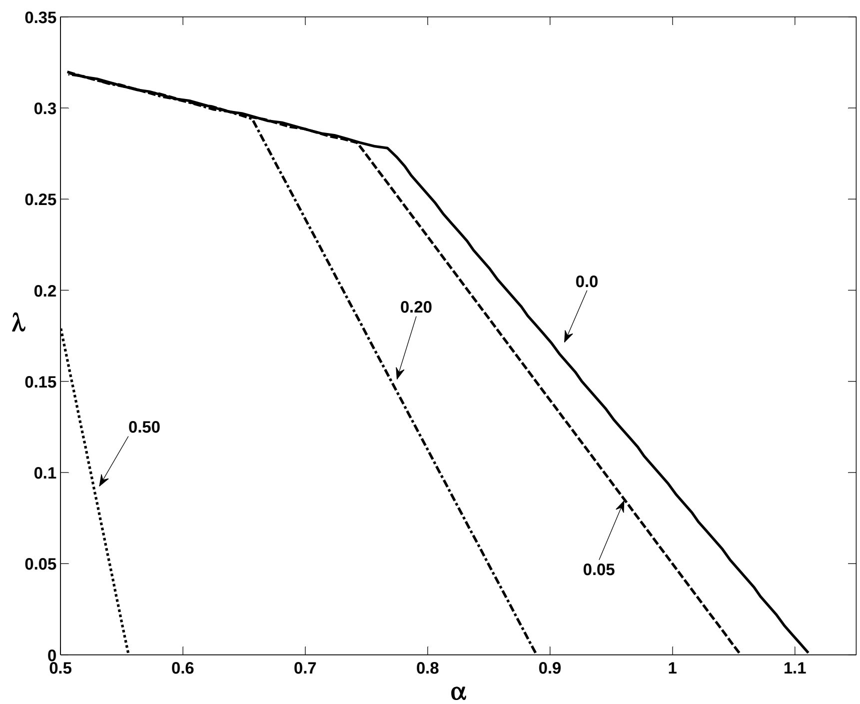
C. Figures and Tables

Figure 1: SG and H–stability



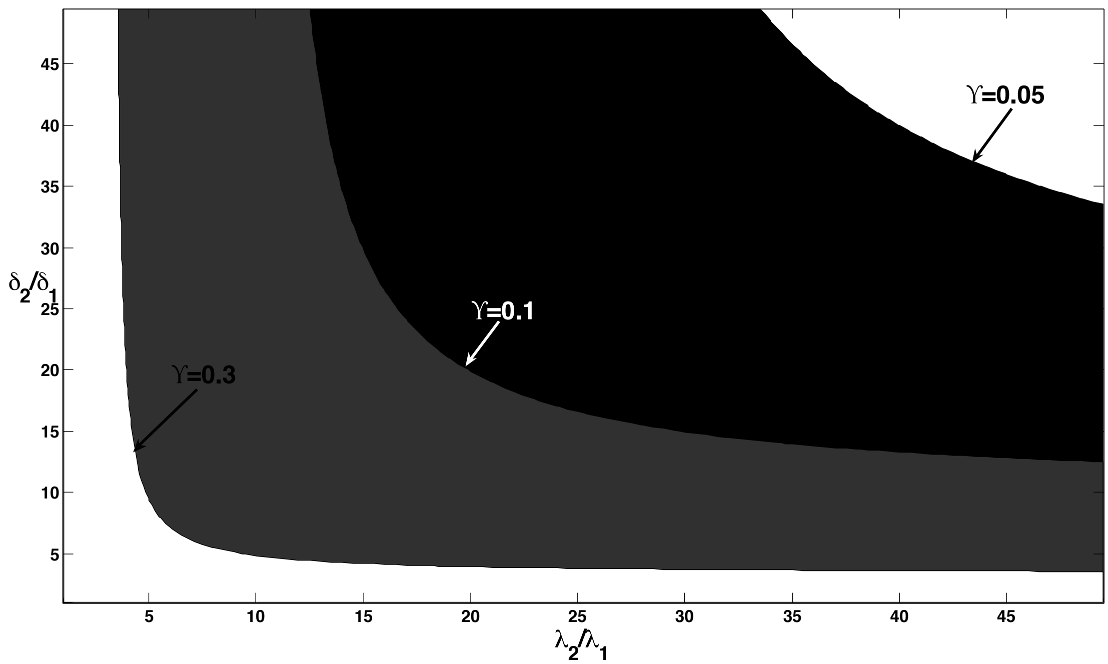
Note: The asterisks represent E–stable equilibria for which the sufficient condition of H–stability is satisfied. The dots show SG–stable equilibria which do not satisfy negative quasi-definiteness. The empty circles are SG-unstable equilibria.

Figure 2: The RLS mean-dynamics convergence speed



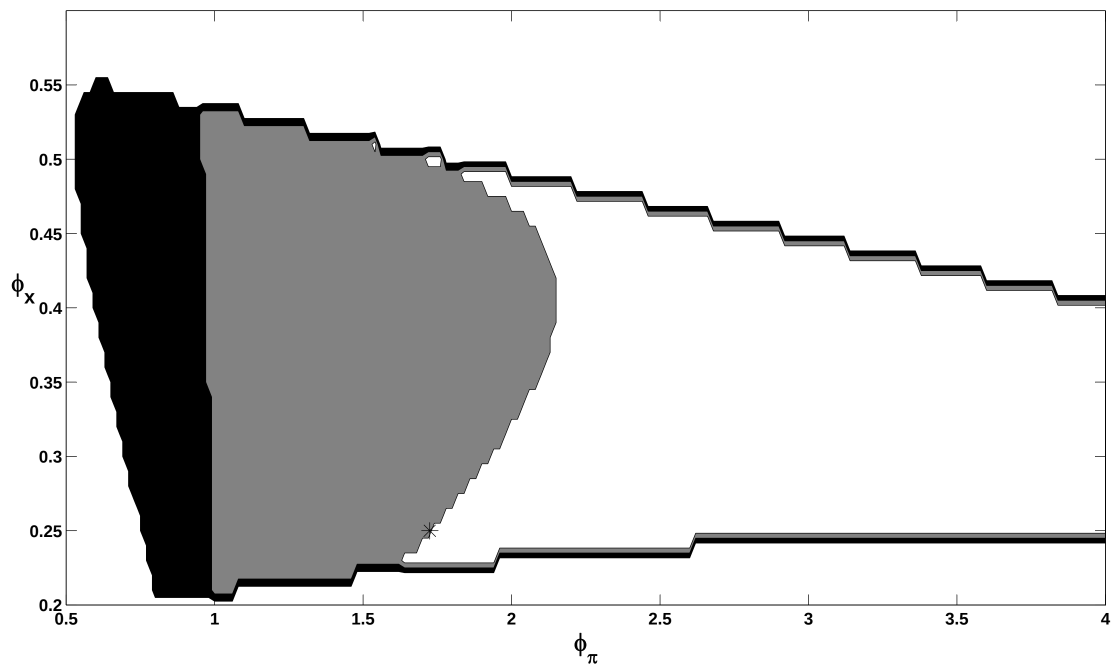
Note: The arrows point to the contour levels of the convergence speed.

Figure 3: The necessary and sufficient conditions Proposition 1



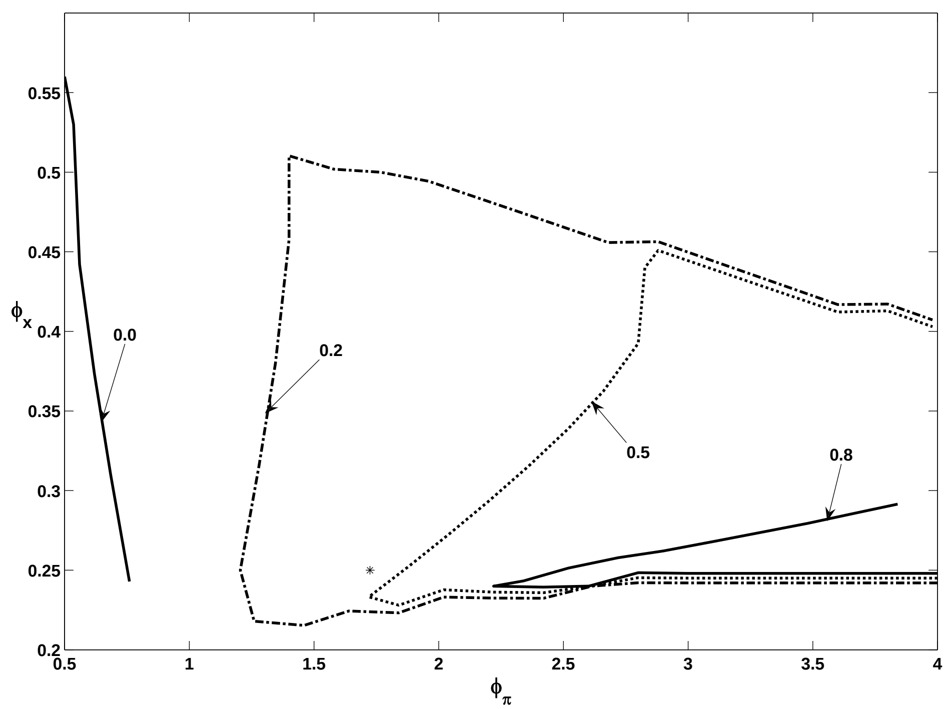
Note: The white area in the upper right corner is where (8) holds for $\Upsilon \geq 0.05$. The black area: (8) holds for $0.10 \geq \Upsilon \geq 0.05$. The grey area: (8) holds for $0.3 \geq \Upsilon \geq 0.10$.

Figure 4: Determinacy and SG–stability: Monetary policy under commitment $\rho = 0.9$



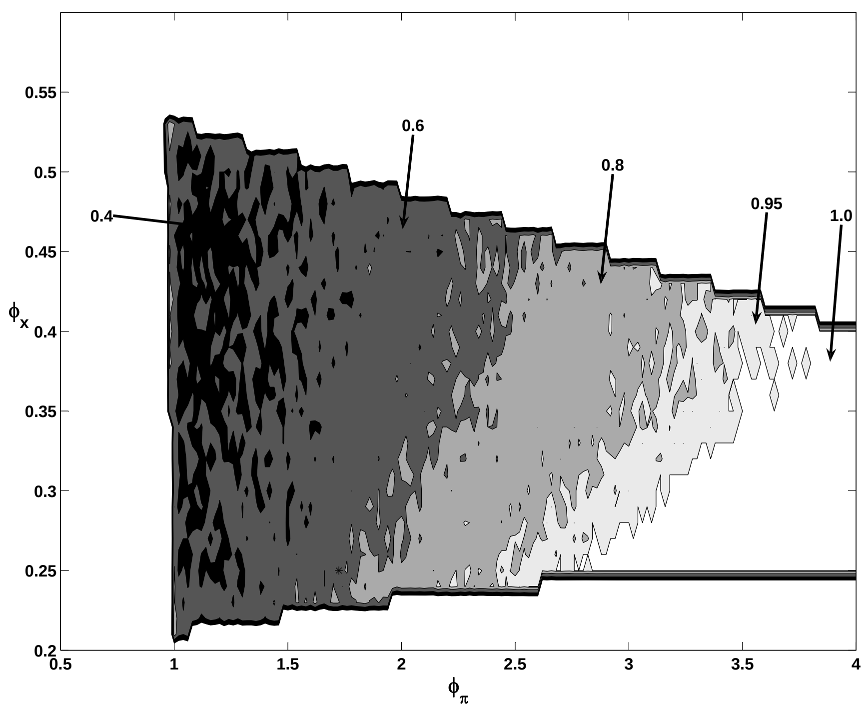
Note: All points within the outer contour are E–stable equilibria. The black area is indeterminate. The grey area is determinate but SG-unstable. The white area is determinate and SG-stable. The asterisks represents the optimal monetary policy under commitment.

Figure 5: The RLS mean-dynamics convergence speed: Monetary policy under commitment $\rho = 0.9$



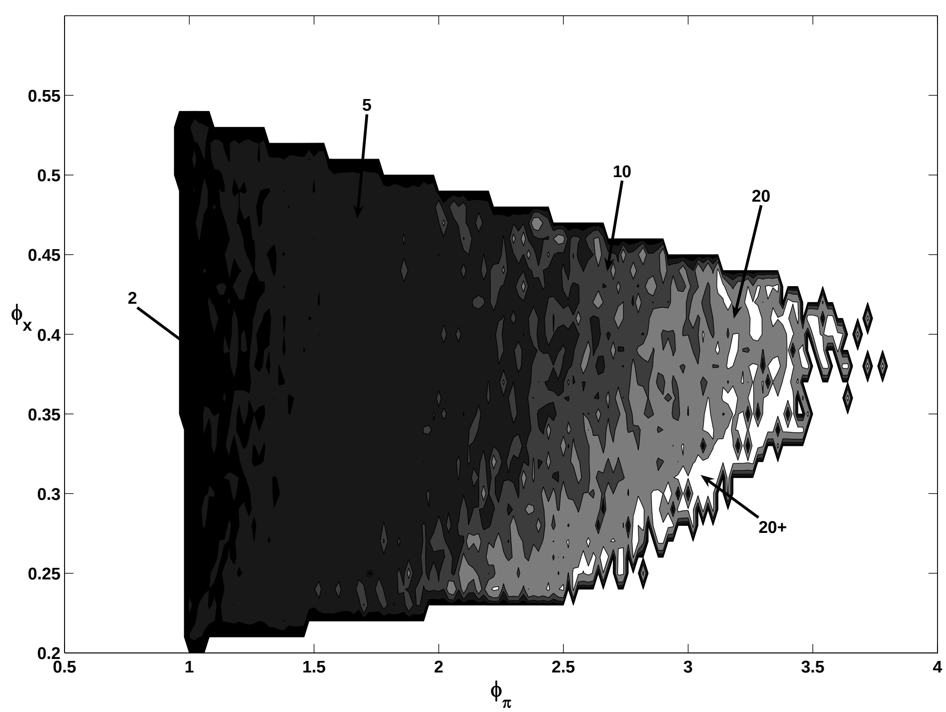
Note: The arrows point to the contour levels of the convergence speed. The asterisks represents the optimal monetary policy under commitment.

Figure 6: The probability of GSG instability: Monetary policy under commitment $\rho = 0.9$



Note: All points within the outer contour are determinate and E–stable equilibria. The probability of GSG instability is at least as large as the arrows indicate.

Figure 7: The minimum measure of misperceptions of GSG instability: Monetary policy under commitment $\rho = 0.9$



Note: All points within the outer contour are determinate and E–stable equilibria. The minimum measure of misperceptions of beliefs to get GSG instability is at least as large as the arrows indicate.

Table 1: Simulations results: Monetary policy under commitment $\rho = 0.9$

	SG-stab	Speed	GSG-stab Prob	Min Dist
$\phi_x = 0.25$				
$\phi_\pi = 1.5$	-	0.33	0.73	2.71
$\phi_\pi = 2.0$	+	0.56	0.83	4.15
$\phi_\pi = 2.5$	+	0.80	0.93	16.00
$\phi_x = 0.35$				
$\phi_\pi = 1.5$	-	0.25	0.64	2.17
$\phi_\pi = 2.0$	-	0.38	0.74	3.59
$\phi_\pi = 2.5$	+	0.48	0.84	6.56
$\phi_x = 0.45$				
$\phi_\pi = 1.5$	-	0.22	0.70	3.79
$\phi_\pi = 2.0$	-	0.33	0.71	5.10
$\phi_\pi = 2.5$	+	0.43	0.76	5.72

Note: SG-stab: SG–stability. Speed: RLS convergence speed. GSG-stab Prob: the probability of GSG–stability. Min Dist: the minimum measure of misperceptions necessary to get GSG instability.

Table 2: Simulations results: Monetary policy under commitment $\rho = 0.6$

		SG-stab	Speed	GSG-stab Prob	Min Dist
$\phi_x = 0.25$					
	$\phi_\pi = 1.5$	+	0.59	1	N/A
	$\phi_\pi = 2.0$	+	0.77	1	N/A
	$\phi_\pi = 2.5$	+	0.97	1	N/A
$\phi_x = 0.35$					
	$\phi_\pi = 1.5$	+	0.51	1	N/A
	$\phi_\pi = 2.0$	+	0.60	1	N/A
	$\phi_\pi = 2.5$	+	0.68	1	N/A
$\phi_x = 0.45$					
	$\phi_\pi = 1.5$	+	0.49	1	N/A
	$\phi_\pi = 2.0$	+	0.57	1	N/A
	$\phi_\pi = 2.5$	+	0.64	1	N/A

Note: SG-stab: SG–stability. Speed: RLS convergence speed. GSG-stab Prob: the probability of GSG–stability. Min Dist: the minimum measure of misperceptions necessary to get GSG instability.

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